Reliability Analysis of Structures under Periodic Proof Tests in Service

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A reliability analysis of structures under random service loads and periodic proof tests is presented. Operational service loads, such as gust loads and maneuver loads, are considered as random processes, and their exceedance curves are used in the analysis. The fatigue process consists of crack initiation, crack propagation, and strength degradation. The time to fatigue crack initiation and the ultimate strength are random variables. While the crack is propagating under service loads, the residual strength decreases progressively, thus increasing the failure rate with time. Proof testing is performed at periodic intervals in service in order to eliminate weak structures, thus increasing structural reliability. When a structure fails under proof testing, a new structure is manufactured and proof-tested for replacement, so that the strength of the structure is renewed. Taking into account all of the random variables, strength degradations, service loads, proof tests, and renewal process of strength due to replacement, the probability of structural failure in service is derived. Numerical examples are worked out to demonstrate the significant influence of the proof load level and the number of periodic proof tests on structural reliability.

I. Introduction

FATIGUE damage is revealed in a structure by the initiation and propagation of visible cracks. The propagation of the fatigue crack leads to the degradation of the residual strength and finally results in a catastrophic failure (e.g., Refs. 1-4). A typical maintenance procedure is to perform scheduled nondestructive inspections. When a fatigue crack is detected during inspection, it may be repaired, depending on the size of the crack, so that both the residual strength and the fatigue strength are renewed, thus increasing the structural reliability. The reliability analyses of aircraft structures with or without scheduled inspections and the optimization of inspection maintenance procedures recently have attracted increasing attention (e.g., Refs. 1-12).

In many practical situations, however, portions of a structure are not accessible for inspection and must go uninspected (e.g., Ref. 2). Other locations, such as butt joints where cracks may occur on an inside surface and are not readily accessible for inspection, may be inspectable only if the structure is torn down. The cost of tear-down maintenance is rather high. As a result, the prevention of failure by scheduled inspection maintenance procedures for these locations may not be feasible economically. In order to prevent failure, repetitive proof testing at regular scheduled intervals becomes a promising alternative procedure and is, in fact, being used currently.[13]

It has been demonstrated that proof testing is a plausible method for eliminating the structures or components containing cracks of intolerable sizes. In fact, proof testing has become an important verification and certification procedure for pressure vessels in the space program (e.g., Refs. 13-18). Although the reliability analysis of structures, typical of spacecraft pressures vessels, under a single proof test prior to service has been investigated (e.g., Refs. 17-18), the fatigue reliability analysis of structures under periodic proof tests has not been available to date. It is the purpose of this paper to present a reliability analysis methodology for fatigue critical structures under periodic proof tests in service. With the present analysis, it is possible to assess quantatively the benefits, in terms of reliability improvement, of periodic proof tests, and to compare the costs and reliability improvement with the scheduled inspections; thereby an optimum maintenance procedure can be determined rationally.

Although the application of the reliability analysis to aircraft structures is emphasized, the approach discussed in this paper is equally applicable to other fatigue-sensitive structures, such as civil and mechanical engineering structures, space shuttles, and marine vehicles. The specific type of service loads considered herein is the flight-by-flight loading to aircraft structures. Failure of structures is assumed to occur when the residual strength is exceeded by service loads. Such a failure mode is referred to as the first-excursion or first-passage failure (e.g., Refs. 19-22). The ultimate strength of the structures is considered a random variable with certain statistical variability.[5]

The fatigue process considered consists of 1) crack initiation, 2) crack propagation, and 3) strength degradation. The time to crack initiation is a random variable.[7,9,11] After the fatigue crack is initiated, crack propagation follows under service loads. While the crack is propagating, the residual strength decreases progressively.[1,7,23-25]

Proof testing is performed at periodic intervals in service. Structures are eliminated or destroyed by the proof test when their residual strength is below the proof load level. Hence, after each proof test, the statistical distribution of the residual strength of the surviving structure is truncated up to the proof load level. Furthermore, the statistical distribution of the residual strength changes with time because of the strength degradation resulting from crack propagation in service. The proof test itself, however, may damage the structure if it is not performed appropriately. To avoid the possible damage, effort should be made to design a proper proof testing procedure, as discussed in Ref. 14. The present reliability analysis, however, can be modified to account for such a possible damage and will be discussed later.

Two different policies are considered, referred to as the renewal policy and the nonrenewal policy, respectively. Under the renewal policy, a new structure or component is manufactured and proof-tested for replacement when a structure fails under the proof test. Thus the strength of the structure after replacement is renewed. Such a renewal process for strength is taken into account in the present analysis. Under the nonrenewal policy, no replacement is made for those structures destroyed by the proof tests, and hence they become unavailable in service.

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Taking into account all of the random variables (strengths), random loads, strength truncations due to proof tests, strength degradations due to service loads, and the renewal processes of strength due to replacement, the solution for the probability of structural failure in service is derived. Numerical results are given to demonstrate the significant influence of the proof load level and the number of periodic proof tests on the reliability of aircraft structures.

II. Service Load and Structural Performance

A. Service Load and Exceedance Curve

Failure of structures may occur when the residual strength is exceeded by the service loads. The critical load that may excees the residual strength level is the gust load for transport-type aircraft, whereas it is the maneuver load for fighter aircraft. Both the exceedance characteristics of the gust load and the maneuver load are discussed in the following sections.

1. Gust exceedance

The expected (average) number of upcrossings of a stress level x by the gust turbulence per flight hour (or flight) can be expressed as[26-28]

$$F_s(x) = N^* \{ u_1 \exp[-(x-x_0)/\sigma_1] + u_2 \exp[-(x-x_0)/\sigma_2] \} \tag{1}$$

in which N^* is the total number of stress cycles (upcrossing over mean stress x_0) per flight hour (or flight), where the mean stress x_0 is the stress due to 1-g loading. The graphical representation of Eq. (1) is referred to as the exceedance curve.[27] Turbulence field parameters u_1, u_2 σ_1, and σ_2 are specified in Ref. 27 for various altitudes.

In practice, the residual strength level is usually much higher than the intensities σ_1 and σ_2. Consequently, it is reasonable to assume that the crossings (or excursions) of the residual strength level are statistically independent. This is referred to as the Poisson approximation in random vibration,[19] and it will be used herein.

Let $R(\tau)$ be the residual strength of a structure at τth flight hour; then the expected number of crossings over the residual strength at τth flight hour is

$$h(\tau) = F_s[R(\tau)] = N^* \sum_{i=1}^{2} u_i \exp\left\{ \frac{-[R(\tau)-x_0]}{\sigma_i} \right\} \tag{2}$$

Because of the Poisson approximation, i.e., independent crossings of the residual strength, $h(\tau)$ is called the failure rate (per flight hour) at τ.

2. Maneuver exceedance

The exceedance curves due to maneuver depend on the mission operational characteristics of fighter aircraft.[29,30] The positive exceedance curve can be approximated by the summation of a finite number of straight lines in the $\log F_s(x)$ vs $(x-x_0)^2$ plot.[31] As a result, the positive exceedance curve can be approximated in analytical form as

$$F_s(x) = N^* \sum_{i=1}^{m} u_i \exp\left\{ \frac{-(x-x_0)^2}{2\sigma_i^2} \right\} \tag{3}$$

in which the number m of straight lines depends on the accuracy required for approximation (see Ref. 31 for detailed discussions). Equation (3) will be used herein expediently for reliability assessment. As a result, the failure rate $h(\tau)$ at τth flight hour for a fighter aircraft follows from Eq. (3) as

$$h(\tau) = N^* \sum_{i=1}^{m} u_i \exp\left\{ \frac{-[R(\tau)-x_0]^2}{2\sigma_i^2} \right\} \tag{4}$$

It should be mentioned that the exceedance curves used in the present analysis, Eq. (1) and (3), refer to the exceedance of *stress* level as specified, e.g., in Refs. 27-29. These exceedance curves are assumed to be available for use in analyses. This should not be confused with the acceleration spectrum.

B. Fatigue Crack Initiation

It is well known that the fatigue process consists of crack initiation, crack propagation, and residual strength degradation. For metallic materials considered herein, the crack initiation stage constitutes a significantly large portion of the fatigue life. The time T_I to crack initiation in service is a statistical variable. Traditionally, the lognormal[1] and the two-parameter Weibull distribution[4,7,11,12] have been used to represent the statistical distribution of T_I. From the analysis of available service data for some types of aircraft, it has been found recently that the statistical distribution of T_I follows other types of distribution.[9] Nevertheless, for the sake of simplicity in presentation, the Weibull distribution is used herein for illustrative purpose:

$$w(t) = \alpha\beta^{-1}(t/\beta)^{\alpha-1}\exp\{-(t/\beta)^{\alpha}\} \tag{5}$$

in which α and β are, respectively, the shape parameter and the scale parameter. From the analysis of extensive coupon data, it was found that $\alpha = 4$ may be appropriate for aluminum materials.[11] The scale parameter β has to be determined from the full-scale test under realistic service flight-by-flight stress spectra (e.g., Refs. 1, 3, 7, 9, and 11).

It should be emphasized that a particular distribution [Eq. (5)] for time to crack initation used herein is not important to the development of the methodology of reliability analysis. It will become apparent later that any type of distribution function can be taken care of in the present analysis.

C. Residual Strength Degradation

Let $a(t)$ and $R(t)$ be the crack size and the residual strength, respectively, at t flight hours *after crack initiation*. For a material specimen, the residual strength $R(t)$ remains the same as the ultimate strength $R(0)$ until the crack size reaches $a_c = a(t_c)$, that is, a material constant,[24] and then it starts to decrease. The crack size $a(t)$ can be obtained by integrating the equation for stable crack propagation,[1-3,6-12,24-25] and the residual strength $R(t)$ $(t > t_c)$ can be obtained from $a(t)$ through the well-known Griffith-Irwin equation for fracture, with the result[24,7]

$$R^c(t) = R^c(0) - \Phi(t-t_c) \tag{6}$$

in which c, Φ, and t_c are constants depending on material properties, crack propagation characteristics, and flight-by-flight service stress spectra.

For complicated structures, such as panels, redundant structures, fail-safe structures, etc., the residual strength of the cracked (damaged) structure depends not only on the crack size but also on the particular design and geometry of the structure. As a result, the residual strength degradation of cracked complicated structures has to be determined from tests or analyses.[1,7,23-25] Testing results presented by Eggwerts[1] for large panels, under flight-by-flight service stress spectra, indicate that the form of Eq. (6) is correct except that the residual strength seems to decrease immediately after a crack is initiated, i.e.,

$$R^c(t_1) = R^c(t_2) - \Phi(t_1 - t_2) \tag{7}$$

in which t_1 and t_2 are the numbers of flight hours *after crack initation*, and $R(t_1)$ and $R(t_2)$ are the residual strengths at t_1 and t_2, respectively. Furthermore, test results given by Eggwerts[1] indicate that $c = 1.0$. Note that the residual strength model for fail-safe structure used in Refs. 7 and 11 corresponds to Eq. (7) with $c=2$. For generality, Eq. (7) will be used in the subsequent derivation for the assessment of structural reliability.

The parameters appearing in Eq. (6) or (7), e.g., c and Φ, can be determined from test results under simulated flight-by-flight random service stress spectra, including ground loads, gust loads, ground-air-ground loads, etc., as presented, e.g., in Ref. 1. The advantage of using simulated random service stress spectra for testing is that the load sequence effects (e.g., retardation) have been accounted for automatically. If test results are not available, a classical approach to estimate c and Φ is to use the method of cycle-by-cycle count,[6,10-12,24] in which case appropriate considerations should be made regarding the load sequence effects, relationships of stress range spectrum to stress spectrum, etc.

D. Distribution and Truncation of Residual Strength under Proof Tests

Let R_0 be the ultimate strength of the structure prior to service and proof testing. Compilation of test results for aircraft structures[5] indicates that the distribution function $F_{R_0}(x)$ of R_0 can be approximated by a two-parameter Weibull distribution:

$$F_{R_0}(x)=P[R_0\leq x]=1-\exp[-(x/\beta_0)^{\alpha_0}] \tag{8}$$

in which α_0 and β_0 are, respectively, the shape parameter and the scale parameter. It is found in Ref. 5 that $\alpha_0=19$ is appropriate for aircraft structures.

Supposing that the proof test is performed at periodic intervals of T_0 flight hours (see Fig. 1), the structure is subjected to the first (or initial) proof testing prior to service. Let $R(0)$ be the structural strength after passing the first proof test at a proof load level r_0. Then, the distribution function $F_{R(0)}(x)$ of $R(0)$ prior to service can be obtained from that of R_0 given by Eq. (8) by a truncation effect as follows[16-18]:

$$F_{R(0)}(x)=1-\frac{P[R_0>x]}{P[R_0>r_0]}=1-\exp\left[\left(\frac{r_0}{\beta_0}\right)^{\alpha_0}-\left(\frac{x}{\beta_0}\right)^{\alpha_0}\right];x\geq r_0 \tag{9}$$

and the probability density function is therefore

$$f_{R(0)}(x)=\frac{\alpha_0}{\beta_0}\left(\frac{x}{\beta_0}\right)^{\alpha_0-1}\exp\left[\left(\frac{r_0}{\beta_0}\right)^{\alpha_0}-\left(\frac{x}{\beta_0}\right)^{\alpha_0}\right];x\geq r_0 \tag{10}$$

The structures that pass the first (or initial) proof test are put into service (Fig. 1). They are referred to as the "original structures." Under the renewal policy, a new structure is manufactured and proof-tested to replace the one destroyed by the proof test. The new structure may fail under the proof test, in which case another structure is manufactured and proof-tested again until one that survives the proof test is obtained. As a result, the strength of the structure after replacement is renewed. Such a strength renewal process will be taken into account later. New structures used for replacement at T_0, $2T_0$,. . ., $(N-1)T_0$ are referred to as the "renewal structures."

After the original structure is put into service, its strength remains the same as $R(0)$ until a fatigue crack is initiated. Let the fatigue crack be initiated at tth flight hour in the first service interval $(0,T_0)$. Then, the distribution of the residual strength of the original structure containing a fatigue crack changes after t flight hours because of 1) strength degradation following Eq. (7), and 2) the truncation effect of proof tests performed at T_0, $2T_0$,. . ., etc. A detail derivation for the change of the statistical distribution of the residual strength of the cracked structure is given in the Appendix. Having survived all of the previous proof tests, the conditional probability density $f_{R(jT_0)}(x)$ of the residual strength $R(jT_0)$ of the original structure right after the $j+1$th proof test can be obtained from $F_{R(jT_0)}(x)$ given by Eq. (A11) in the Appendix, with the result

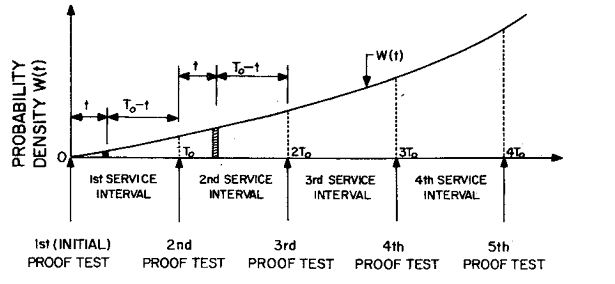


Fig. 1 Periodic proof tests, service intervals, and probability density for crack initiation.

$$f_{R(jT_0)}(x)=\left(\frac{\alpha_0}{\beta_0}\right)\left(\frac{x}{\beta_0}\right)^{c-1}\left\{\frac{x^c+\Phi(jT_0-t)}{\beta_0{}^c}\right\}^{\alpha_0/c-1}$$
$$\times\exp\left\{\left[\frac{r_0{}^c+\Phi(jT_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}-\left[\frac{x^c+\Phi(jT_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}\right\};x\geq r_0 \tag{11}$$

in which

$$f_{R(jT_0)}(x)=\mathrm{d}F_{R(jT_0)}(x)/\mathrm{d}x$$

is a conditional density function, the condition being that the fatigue crack is initiated at tth flight hour in $(0, T_0)$.

A quantity of importance in the reliability assessment is the probability that the structure will be destroyed under each proof test. Let $B_j^*(t)$ be the conditional probability that an original structure will survive up to $j+1$ proof tests at jT_0 flight hours. Furthermore, let $B_j(t)$ be the conditional probability that the original structure will fail during the $j+1$th proof test at jT_0. Both $B_j^*(t)$ and $B_j(t)$ are conditional probabilities under the conditions that a fatigue crack is initiated at tth flight hour in $(0,T_0)$ and that the original structure does not fail in service before jT_0. Since the probability of being destroyed by the $j+1$th proof test is the difference between the probability of surviving $j+1$ proof tests and the probability of surviving j proof tests, it is obvious that

$$B_j(t)=B_{j-1}^*(t)-B_j^*(t) \tag{12}$$

in which $B_0^*(t)=1.0$.

The probability $B_j^*(t)$ of surviving up to the $j+1$th proof test performed at jT_0 for an original structure has been derived in the Appendix, with the result

$$B_j^*(t)=\exp\left\{\left(\frac{r_0}{\beta_0}\right)^{\alpha_0}-\left(\frac{r_0{}^c+\Phi(jT_0-t)}{\beta_0{}^c}\right)^{\alpha_0/c}\right\} \tag{13}$$

III. Conditional Probability of Failure

The following formula for the conditional probability will be used frequently:

$$P[A]=\int_{-\infty}^{\infty}P[A\,|\,X=x]f_X(x)\mathrm{d}x \tag{14}$$

in which $P[A]$ is the probability of failure, $P[A\,|\,X=x]$ is the probability of failure given (or under the condition) that the random variable X is equal to a value x, and $f_X(x)$ is the probability density of X. Furthermore, if the total failure rate in one service interval is H, then the probability of failure P_f in that service interval is

$$P_f = 1 - \exp(-H) \tag{15}$$

A. Transport-Type Aircraft

Supposing that the crack is initiated at tth flight hour, the strength before crack initiation in $(0,t)$ is $R(0)$, and hence the conditional failure rate given that $R(0)=x$ follows from Eqs. (2) and (1) as $h_0[R(0)=x]=F_s(x)$. Hence, $F_s(x)$ is the conditional failure rate before crack initiation given that $R(0)=x$.

Substituting Eq. (7) with $t_1=t+\tau, t_2=t$, i.e.,

$$R(t+\tau)=[R^c(0)-\Phi\tau]^{1/c}$$

into Eq. (2), the conditional failure rate at $t+\tau$, given that $R(0)=x$ and that a crack is initiated at t such that $R(t)=R(0)$, is obtained

$$h_1[\tau|R(0)=x]=N^*\sum_{i=1}^{2}u_i\exp\left\{\frac{-[(x^c-\Phi\tau)^{1/c}-x_0]}{\sigma_i}\right\} \tag{16}$$

The total conditional failure rate in $(0,T_0)$, denoted by $H_1[t|R(0)=x]$, under the condition that $R(0)=x$ and that the crack is initiated at t, is obtained by the integration of $F_s(x)$ from 0 to t plus the integration of h_1 [Eq. (16)] from 0 to T_0-t:

$$H_1[t|R(0)=x]=F_s(x)t+H_1^*[t|R(0)=x] \tag{17}$$

in which

$$H_1^*[t|R(0)=x]=N^*\sum_{i=1}^{2}u_i\int_0^{T_0-t}\exp\left\{\frac{-[(x^c-\Phi\tau)^{1/c}-x_0]}{\sigma_i}\right\}\mathrm{d}\tau \tag{18}$$

For $c=1$, the closed-form integration is possible:

$$H_1^*[t|R(0)=x]=N^*\sum_{i=1}^{2}\frac{u_i\sigma_i}{\Phi}\exp\left\{\frac{-(x-x_0)}{\sigma_i}\right\}\left\{\exp\left[\frac{\Phi(T_0-t)}{\sigma_i}\right]-1\right\} \tag{19}$$

Hence, the conditional failure probability in the first service interval follows from Eq. (15) as $1-\exp\{-H_1[t|R(0)=x]\}$, and the probability of failure in the first service interval $(0,T_0)$, denoted by $p_1^*(t)$, can be obtained using Eq. (14) as

$$p_1^*(t)=\int_{r_0}^{\infty}f_{R(0)}(x)\left\{1-\exp\{-H_1[t|R(0)=x]\}\right\}\mathrm{d}x \tag{20}$$

in which $f_{R(0)}(x)$ is the probability density of $R(0)$ given by Eq. (10), and $H_1[t|R(0)=x]$ is given by Eq. (17).

Let $h_{j+1}[\tau|R(jT_0)=x]$ be the conditional failure rate at $jT_0+\tau$ for the original structures that have survived $j+1$ proof tests. The residual strength $R(jT_0+\tau)$ at $jT_0+\tau$ is related to $R(jT_0)$ through Eq. (7), i.e., $R^c(jT_0+\tau)=R^c(jT_0)-\Phi\tau$. Substituting this relationship into Eq. (2) and comparing the result with Eq. (16), one obtains

$$h_{j+1}[\tau|R(jT_0)=x]=N^*\sum_{i=1}^{2}u_i\exp\left\{\frac{-[(x^c-\Phi\tau)^{1/c}-x_0]}{\sigma_i}\right\}=h_1[\tau|R(0)=x] \tag{21}$$

The total conditional failure rate in the $j+1$th service interval $[jT_0, (j+1)T_0]$, denoted by $H_{j+1}[R(jT_0)=x]$, is the integral of Eq. (21) with respect to τ:

$$H_{j+1}[R(jT_0)=x]=\int_0^{T_0}h_{j+1}[\tau|R(jT_0)=x]\,\mathrm{d}\tau=H_1^*[0|R(0)=x] \tag{22}$$

where $H_1^*[0|R(0)=x]$ is given by Eq. (18) with $t=0$. Hence, for $c=1$,

$$H_{j+1}[R(jT_0)=x]=H_1^*[0|R(0)=x]$$

is given by Eq. (19) with $t=0$.

The probability of failure $p_{j+1}(t)$ in the $j+1$th service interval $[jT_0, (j+1)T_0]$ for the original structures is obtained from Eqs. (22) and (18) by the application of Eqs. (14) and (15), with the result

$$p_{j+1}^*(t)=\int_{r_0}^{\infty}f_{R(jT_0)}(x)\left\{1-\exp\{-H_{j+1}[R(jT_0)=x]\}\right\}\mathrm{d}x \tag{23}$$

in which $f_{R(jT_0)}(x)$ is the probability density function of $R(jT_0)$ given by Eq. (11), and $H_{j+1}[R(jT_0)=x]$ is given by Eqs. (22, 18, and 19).

B. Fighter Aircraft

For fighter aircraft where the exceedance of the residual strength is due to maneuver loads, the conditional probability of failure in each service interval, $p_j^*(t)$ for $j=1, 2, \ldots$, derived in Eqs. (20) and (23) for transport-type aircraft still holds except that Eq. (3) should be used for $F_s(x)$, and Eqs. (16) and (18). Hence, for $c=1$,

$$H_1^*[t|R(0)=x]=N^*\sum_{i=1}^{m}\frac{\sqrt{\pi}u_i\sigma_i}{\sqrt{2}\Phi}\left\{\mathrm{erf}\left(\frac{x-x_0}{\sqrt{2}\sigma_i}\right)-\mathrm{erf}\left(\frac{x-x_0-\Phi(T_0-t)}{\sqrt{2}\sigma_i}\right)\right\} \tag{24}$$

in which erf( ) is the error function.

It may be worthwhile to reiterate that the quantities

$$h_{j+1}[\tau|R(jT_0)=x]=h_1[\tau|R(0)=x], H_1[t|R(0)=x], H_1^*[t|R(0)=x], H_{j+1}[R(jT_0)=x]=H_1^*[0|R(0)=x]$$

and $p_{j+1}^*(t)$ derived in Eqs. (16-24) are dependent on the condition that a fatigue crack is initiated at tth flight hours, although such an implied condition is not identified specifically in their expressions because of convenience in presentation.

IV. Probability of Failure under Periodic Proof Tests

A. Renewal Policy

Having obtained the conditional probability of failure in each service interval $p_j^*(t)$ for the surviving (original) structures, under the condition that the crack is initiated at t, we are in the position to derive the unconditional failure probability in each service interval not only for the original structures but also for the renewal structures. Again, Eqs. (14) and (15) will be used to transform the conditional failure probability into the unconditional failure probability.

The probability of failure $p(1)$ in the first service interval $(0,T_0)$ can be written in the form of Eq. (14):

$$p(1) = \int_0^{T_0} p_1^*(t)w(t)\mathrm{d}t + \int_{T_0}^{\infty} Vw(t)\mathrm{d}t \tag{25}$$

in which $w(t)$ is the probability density function of time to crack initiation given by Eq. (5), and V is the probability of failure in $(0,T_0)$ under the condition that the crack is initiated after T_0. The first term in Eq. (25) is the probability of failure when the crack is initiated in $(0,T_0)$, and the second term is the failure probability when the crack is initiated after T_0.

When the fatigue crack is initiated after T_0, the conditional failure rate in $(0,T_0)$ is constant and is equal to $h_0[R(0)=x] = F_s(x)$ given by Eq. (1). Applications of Eqs. (14) and (15) yields

$$V = \int_{r_0}^{\infty} f_{R(0)}(x)\{1-\exp[-F_s(x)T_0]\}\mathrm{d}x \tag{26}$$

which is not a function of t.

Substitution of Eqs. (5) and (26) into Eq. (25) leads to the following:

$$p(1) = Z_1 + \int_0^{T_0} p_1^*(t)w(t)\mathrm{d}t \tag{27}$$

where

$$Z_1 = V\exp\{-(T_0/\beta)^{\alpha}\} \tag{28}$$

is the probability of failure in $(0,T_0)$ when the crack is initiated after T_0.

Let Z_j be the probability of failure in the jth service interval $[(j-1)T_0, jT_0]$ under the condition that the crack is initiated after jT_0. Then, similarly to Eq. (28), it easily can be shown that

$$Z_j = \int_{jT_0}^{\infty} Vw(t)\mathrm{d}t = V\exp\left\{-\left(\frac{jT_0}{\beta}\right)^{\alpha}\right\} \tag{29}$$

The probability of failure in the second service interval $(T_0,2T_0)$, denoted by $p(2)$, is

$$p(2) = Z_2 + \int_0^{T_0} p_1^*(t)w(T_0+t)\mathrm{d}t + \int_0^{T_0} q_{12}(t)w(t)\mathrm{d}t \tag{30}$$

in which the first term is contributed by the event of crack initiation after $2T_0$ given by Eq. (29), the second term is contributed by the event of crack initiation in the second service interval, and the third term is contributed by the event of crack initiation in the first service interval. Note that the probability of failure in the $j+1$th service interval $(j=1,2,...)$ under the condition that the crack is initiated at jT_0+t in that interval (see Fig. 1) can be shown to be equal to $p_1^*(t)$. Hence $p_1^*(t)$ is used in the second term of Eq. (30).

In Eq. (30), $q_{12}(t)$ is the probability of failure in $(T_0,2T_0)$ under the condition that the crack is initiated at t in the first service interval $(0,T_0)$, which consists of two parts:

$$q_{12}(t) = B_1(t)p(1) + B_1^*(t)p_2^*(t) \tag{31}$$

in which the second term is attributed to the original structure with surviving probability $B_1^*(t)$ at T_0 under the second proof test, and $p_2^*(t)$ is the conditional failure probability as derived in Eq. (23). The first term in Eq. (31) is the contribution from the renewal structure manufactured at T_0 when the original structure fails at T_0 with probability $B_1(t)$, and the probability of failure of the renewal structure in $(T_0,2T_0)$ is $p(1)$, preferred to as the renewal failure probability. Note that $B_1(t) + B_1^*(t) = 1.0$.

It is noted that $B_1^*(t)$ and $B_1(t)$ are conditional probability under the condition that the structure survives random service loads in the time interval $(0,T_0)$. For consistency in approximation, however, such a condition has been removed in deriving Eq. (31) for $q_{12}(t)$, because the Poisson approximation[19-22] has been used in the present analysis in evaluating the service failure probability [see Eqs. (2) and (4)]. The derivation of Eqs. (34) and (35) will have the same implication for $B_j^*(t)$ and $B_j(t)$. further discussion in this regard will be made in Sec. VI.

The probability of failure $p(3)$ in the third service interval $(2T_0,3T_0)$ is contributed by the following events: 1) the crack is initiated after $3T_0$, 2) the crack is initiated in $(2T_0,3T_0)$, 3) the crack is initiated in $(T_0, 2T_0)$ and 4) the crack is initiated in $(0,T_0)$. These contributions are given, respectively, as follows:

$$p(3) = Z_3 + \int_0^{T_0} p_1^*(t)w(2T_0+t)\mathrm{d}t + \int_0^{T_0} q_{23}(t)w(T_0+t)\mathrm{d}t + \int_0^{T_0} q_{13}(t)w(t)\mathrm{d}t \tag{32}$$

in which $q_{23}(t)$ is the probability of failure in the third service interval $(2T_0,3T_0)$ under the condition that the crack is initiated at T_0+t in the second service interval $(T_0,2T_0)$, and $q_{13}(t)$ is the probability of failure in $(2T_0,3T_0)$ under the condition that the crack is initiated at t in $(0,T_0)$.

It easily can be shown that $q_{23}(t)$ is equal to $q_{12}(t)$ given by Eq. (31), i.e.,

$$q_{23}(t) = q_{12}(t) \tag{33}$$

The probability of failure $q_{13}(t)$ in the third service interval $(2T_0,3T_0)$ under the condition that the crack is initiated at t in the first service interval $(0,T_0)$ consists of three parts:

$$q_{13}(t) = B_1(t)p(2) + B_2(t)p(1) + B_2^*(t)p_3^*(t) \tag{34}$$

which is self-explanatory. The first term is attributed to the renewal structure manufactured at T_0 where the renewal probability is $B_1(t)$ and the renewal failure probability in $(2T_0,3T_0)$ is $p(2)$, given by Eq. (30). The second term is attributed to the renewal structure manufactured at $2T_0$ to replace the original structure, which fails under the proof test performed at $2T_0$. The probability of such a replacement is $B_2(t)$ and the renewal failure probability is $p(1)$. The third term is attributed to the original structure, where $B_2^*(t)$ is the probability of surviving all of the proof tests up to $2T_0$, and the probability of failure in $(2T_0,3T_0)$ for such an original structure is $p_3^*(t)$ derived in Eq. (23). It is important to note that, in Eq. (34), $B_1(t) + B_2(t) + B_2^*(t) = 1.0$.

In a similar fashion, the general solution for the probability of failure in the jth service interval $[(j-1)T_0, jT_0]$ can be derived as follows:

$$p(j) = Z_j + \sum_{k=1}^{j} \int_0^{T_0} G_{j-k+1}(t)w[(k-1)T_0+t]\mathrm{d}t \tag{35}$$

in which Z_j and V are given by Eqs. (26) and (29), respectively, and

$$G_j(t) = B_{j-1}^*(t)p_j^*(t) + \delta_{j-1}\sum_{i=1}^{j-1} B_{j-i}(t)p(i) \tag{36}$$

where $p_j^*(t)$ is given by Eqs. (20) and (23), and $B_j(t)$ and $B_j^*(t)$ are given by Eqs. (12) and (13) with $B_0^*(t) = 1$. In Eq. (36), $\delta_{j-1} = 0$ for $j-1 \leq 0$, and $\delta_{j-1} = 1$ for $j-1 > 0$. The solution derived in Eq. (35) holds for both transport-type aircraft and fighter aircraft, except that for each case appropriate $F_s(x)$ and $H_1^*[t|R(0)=x]$ given in the previous section should be used.

B. Nonrenewal Policy

When the original structures fail under the proof tests, they are not replaced by new structures, and hence the aircraft becomes unavailable for service. Under such a nonrenewal policy, the solution derived in Eqs. (35) and (36) still holds except that the second term (summation terms) appearing in Eq. (36) should be disregarded, since it represents the contribution of failure probability from the renewal structures.

C. Cumulative Probability of Failure and Fleet Size

Let $P_f(r_0,j)$ be the cumulate probability of failure in j service intervals $(0,jT_0)$. It can be shown that[9]

$$P_f(r_0,j) = 1 - \prod_{i=1}^{j} [1-p(i)] \tag{37}$$

Equation (37) indicates that the survival in $(0, jT_0)$ implies the survival in each service interval up to jT_0. The probability of failure $P_f(r_0, j)$ in $(0, jT_0)$ derived in Eq. (37) holds for a single airplane. For a fleet of M airplanes, the probability of first failure in $(0, jT_0)$, denoted by $P_M(r_0, j)$, is[9]

$$P_M(r_0,j) = 1 - [1-P_f(r_0,j)]^M$$

D. Probability of Failure without Proof Test

Let $P(t^*)$ be the cumulative probability of failure in the service interval $(0,t^*)$ for structures without the application of proof tests. The solution $P(t^*)$ can be derived easily in a manner similar to Eq. (25) as follows:

$$P(t^*) = \int_0^{t^*} \bar{p}_I(t)\, w(t)\,\mathrm{d}t + \int_{t^*}^{\infty} V w(t)\,\mathrm{d}t \tag{38}$$

$$\bar{p}_I(t) = \int_{r_0}^{\infty} f_{R_0}(x)\{1-\exp(-H_I[t|R_0=x])\}\mathrm{d}x \tag{39}$$

in which $H_I[t|R_0=x] = H_I[t|R(0)=x]$ is given by Eqs. (17) and (18), with T_0 being replaced by t^*. $f_{R_0}(x)$ is density function of the ultimate strength R_0, obtained by differentiating Eq. (8):

$$f_{R_0}(x) = (\alpha_0/\beta_0)(x/\beta_0)^{\alpha_0 - 1}\exp[-(x/\beta_0)^{\alpha_0}] \tag{40}$$

V. Numerical Examples

A. Example 1: Transport-Type Aircraft

A critical component of a transport-type aircraft wing is considered to demonstrate the approach proposed in this study. The parameters associated with the exceedance curve of the design gust spectrum [Eq. (1)] are as follows: $u_1 = 99.5\%$, $u_2 = 0.5\%$, $\sigma_1 = 0.07$ g, $\sigma_2 = 0.18$ g, where the mean stress $x_0 = 1$ g $= 10$ ksi. This is the same gust spectrum used in Ref. 7. It is assumed that each flight is of 2 hr duration, and in one flight hour the structure is subjected to 600 gust cycles, i.e., $N^* = 600$. The crack size at crack initiation, is $a_0 = 0.4$ in. The shape parameter for time to crack initiation is $\alpha = 4$, and the scale parameter is $\beta = 30{,}000$ flight hours [Eq. (5)]. The material of the critical component has a Weibull distributed ultimate strength, with the shape parameter $\alpha_0 = 19$,[5] and the scale parameter (characteristic strength) assumed to be $\beta_0 = 57$ ksi [Eq. (8)]. After crack initiation, the residual strength decreases linearly with respect to the service hour as demonstrated by test results of Ref. 1, i.e., $c = 1$ in Eq. (7). It is assumed that after crack initiation the characteristic strength $\beta_0 = 57$ ksi reduces to 20 ksi in 5000 flight hours, i.e., $\Phi = 0.0074$ ksi per flight hour [see Eq. (7)]. The design service life T for the airplane is 15,000 flight hours.

With all of the input parameters just given, the conditional failure rates $h_0[R(0)=x]$ and $H_I[t|R(0)=x]$ are computed from Eqs. (1, 17, and 18). Then, the conditional probability of failure $p_j^*(t)$ in each service interval, $j = 1, 2, \ldots, N$, is estimated using Eqs. (20-23). Finally, the unconditional failure probability $p(j)$ in each service interval and the cumulative probability of failure $P_f(r_0,j)$ in j service intervals $(0,jT_0)$ are computed from Eqs. (35-37).

Results for the cumulative probability of failure $P_f(r_0, j)$ [Eq. (37)] for one airplane are plotted in Fig. 2 as a function of service hours for various values of proof load level r_0 and various numbers N of periodic proof tests in the design service life of 15,000 flight hours. For instance, the number of proof tests $N = 15$ in Fig. 2 indicates that proof testing is performed at every 1000 flight hours, i.e., $T_0 = 1000$ flight hours. Also plotted in Fig. 2 as a dashed curve is the cumulative probability of failure without proof testing, i.e., $N = 0$.

It is noticed that all of the curves in Fig. 2 consist of two segments with completely different characteristics. The first segment is within 4000 flight hours, which is attributed essentially to the exceedance of gust loads over the ultimate strength $R(0)$ and is referred to as the chance failure (see Ref. 7). The second segment in the region greater than 4000 flight hours is essentially due to the exceedance of gust loads over the residual strength, and hence the failure rate increases with respect to the service hour, which is typical of progressive damage, and is referred to as the fatigue failure mode (see Ref. 7 for detailed discussions).

Figure 2 clearly demonstrates that significant improvement on structural reliability can be achieved by the application of periodic proof tests and that the reliability increases as the proof load level r_0 or the number N of periodic proof tests increases. Note that the purpose of periodic proof tests is to eliminate the structures having residual strengths below the proof load level. Hence, the ultimate benefit that one can achieve through periodic proof tests is to maintain the residual strength of the structure above the proof load level in service, i.e., to maintain the crack size smaller than the critical crack size associated with the proof load level. Such an ultimate reliability improvement is shown, for instance, in Fig. 2a by the curve associated with $N = 20$. The number N of periodic proof tests beyond this limit results in practically no further benefit, such as the curve associated with $N = 20$ in Fig. 2a. The existence of such a limiting improvement can be observed clearly in Fig. 2.

The probability of failure $P_f(r_0,N)$ within the design service life of 15,000 flight hours is summarized in Fig. 3 for various proof load levels. It is observed from Fig. 3 that, associated with each proof load level, there is a limit on the number N of periodic proof tests beyond which the reliability improvement vanishes, for instance, $N = 15$, $r_0 = 0.95\beta_0$. As a result, associated with each proof load level r_0 there is a limiting reliability beyond which improvement is impossible to achieve, as indicated by the asymptote of each curve. For instance, it is impossible to achieve a failure probability smaller than 10^{-5} for $r_0 = 0.7\beta_0$, regardless of the number of periodic proof tests.

B. Example 2: Fighter Aircraft

A critical component of a fighter aircraft wing subjected to F-111 maneuver spectra[29] is considered. The exceedance curve of maneuver loads per flight (3-hr duration) can be written analytically in the form of Eq. (3), with $m = 2$, $N^*u_1 = 15.8$, $N^*u_2 = 158$, $\sigma_1 = 2.405$ $g = 8.417$ ksi, $\sigma_2 = 1.171$ $g = 4.1$ ksi, $x_0 = 1$ g $= 3.5$ ksi. The shape parameter for time to fatigue crack initiation is $\alpha = 4$, and the scale parameter is $\beta = 3000$ flights [Eq. (5)]. The statistical distribution of the ultimate strength of the component is Weibull, with the shape parameter $\alpha_0 = 15$ and the scale parameter $\beta_0 = 65$ ksi [Eq. (8)]. After crack initiation, the residual strength decreases linearly with respect to the number of flights,[1] i.e., $c = 1$ in Eq. (7). It is assumed that, after a crack is initiated at 0.04 in., the characteristic strength β_0 reduces to $0.45\beta_0$ within 500 flights, i.e., $\Phi = 0.0715$ ksi/flight [Eq. (7)]. The design service life T for the aircraft is 1500 flights.

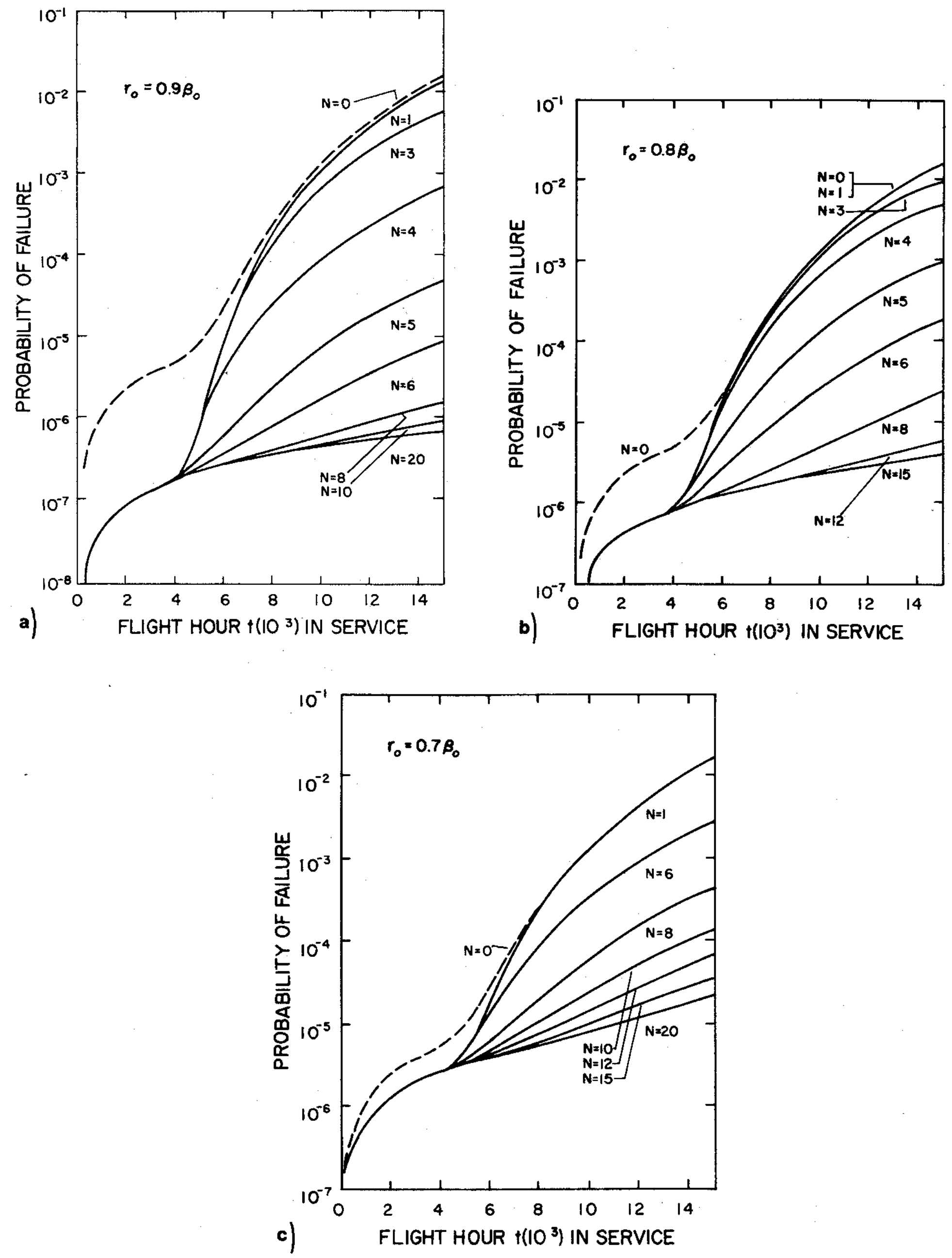


Fig. 2 Probability of failure vs service time ($\beta_0 = 57$ ksi). a) $r_0 = 0.9\beta_0$. b) $r_0 = 0.8\beta_0$. c) $r_0 = 0.7\beta_0$.

Results for the cumulative probability of failure $P_f(r_0, j)$ [Eq. (37)] in $(0, jT_0)$ for one airplane are plotted in Fig. 4 as a function of service flights. Also plotted in Fig. 4 as a dashed curve is the cumulative probability of failure without proof testing, i.e., $N=0$. The probability of failure $P_f(r_0,N)$ within the design service life of 1500 flights is summarized in Fig. 5 for various values of proof load level r_0.

Similar conclusions as in example 1 for transport-type aircraft have been observed from Figs. 4 and 5: 1) significant improvement on structural reliability can be achieved by the application of periodic proof tests; 2) the reliability increases as the proof load level r_0 or the number N of proof tests increases; and 3) associated with each proof load level there is a limiting reliability improvement, beyond which it is impossible to achieve, regardless of the number N of proof tests.

VI. Conclusion and Discussion

An exploratory reliability analysis of metallic structures under random service loads and periodic proof tests has been presented. The reliability of structures is obtained as a function of the statistics of ultimate strength, exceedance statistics of service loads, residual strength degradation characteristics, design mean stress, proof load level, number of periodic proof tests, etc. It has been demonstrated that the reliability of structures can be improved significantly by the application of periodic proof tests.

It has been mentioned previously that the proof test itself may damage the structure, if it is not performed appropriately. To design a proper proof testing procedure, effort is needed along the lines discussed, for instance, in Ref. 14. If such an undesirable damaging effect cannot be avoided completely under some particular situations, it can be accounted for in the present analysis with a modification. The modification can be made by introducing a damaging factor immediately after each truncation (due to proof testing) of the distribution function of strength, e.g., Eq. (9). The damaging factor should reflect the crack opening effect during unloading of proof testing (see Ref. 14) for each particular situation.

In the reliability analysis for fighter aircraft, the positive maneuver exceedance curve is approximated by Eq. (3) using

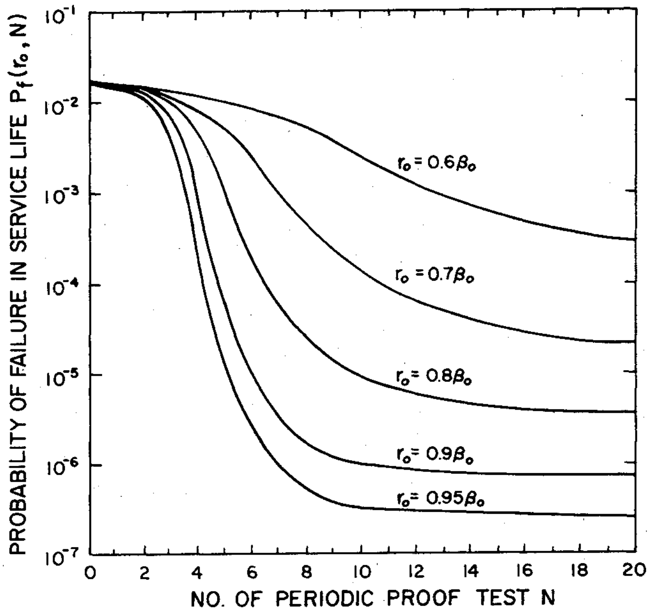


Fig. 3 Probability of failure in design service life T vs number of proof tests N and proof load level r_0.

the method of Press.[31] Although such an expedient approach simplifies the analyses, it may be too conservative. Since the maneuver operation is controlled, it is believed that there is an upper load level beyond which the exceedance will not occur. Hence, a refined model for maneuver exceedance is to set an upper bound, i.e., cutoff exceedance level, denoted by x^*, beyond which the probability of exceedance is zero. The solution derived in this paper still holds for such a refined model, except that the upper limit of integration appearing in Eqs. (20, 23, and 26), which is ∞, should be replaced by x^*.

The time to fatigue crack initiation has been assumed to follow a two-parameter Weibull distribution [Eq. (5)]. It implies that the new components (or structures) have no initial damage, i.e., detectable crack. In reality, however, new components do have certain probability of containing initial detectable cracks (e.g., Ref. 10), and it should be accounted for. Such a situation, however, is a special case in the present analyses. The solution easily can be derived 1) by setting the time to crack initiation equal to zero, and 2) by replacing the ultimate strength distribution $F_{R_0}(x)$ [Eq. (8)] by a residual strength distribution that is derived from the initial crack size distribution. Then, it can be shown that the solution is much simpler than that of the situation considered in this paper.

The analysis of structural reliability presented herein (also in Ref. 7) is based on Poisson approximation. Let $v(\tau)d\tau$ be the probability of exceeding the residual strength $R(\tau)$ by the random service stress $S(\tau)$ in $(\tau, \tau+d\tau)$, under the condition that the structure has survived in service up to τ:

$$v(\tau)d\tau = p[R(\tau) \leq S(\tau) \mid R(t) > S(t); 0 \leq t < \tau] \tag{41}$$

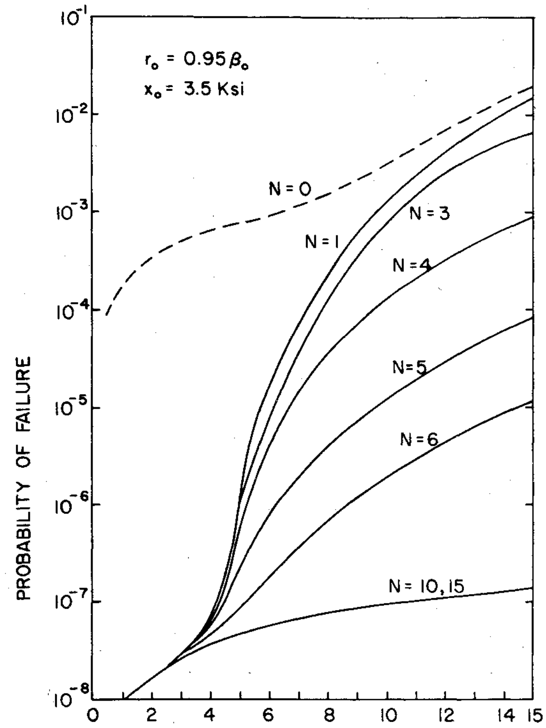
$v(\tau)$ is called failure rate (or risk function) at τ. With Poisson approximation, one obtains, for $d\tau$ equals to one flight hour,

$$v(\tau) \simeq p[R(\tau) \leq S(\tau)] = h(\tau) = F_s[R(\tau)] \tag{42}$$

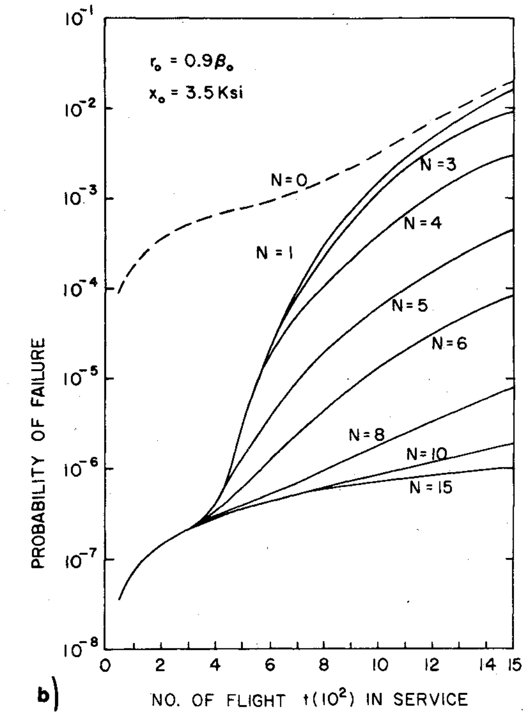
in which $h(\tau)$ is given by Eqs. (2) and (4).

This approximation implies that the probability of failure in service before τ is negligibly small, so that the *conditional* event

$$\{R(t) > S(t); 0 \leq t < \tau\}$$



a)



b)

Fig. 4 Probability of failure vs service time (fighter aircraft $\beta_0 = 65$ ksi). a) $r_0 = 0.95\beta_0$. b) $r_0 = 0.9\beta_0$.

is almost a sure event and can be removed. As such, the condition that the structure has survived in service up to jT_0.

$$\{R(\tau) > S(\tau); 0 \leq \tau < jT_0\}$$

associated with $B_j(t)$ and $B_j^*(t)$ is removed in deriving Eqs. (31-36) to be consistent with the framework of Poisson approximation. It should be noted that not only the general solutions derived in Eqs. (31-36) but also Eq. (37) involve Poisson approximation. The Poisson approximation is conservative in estimating structural reliability[19-22] and is realistic

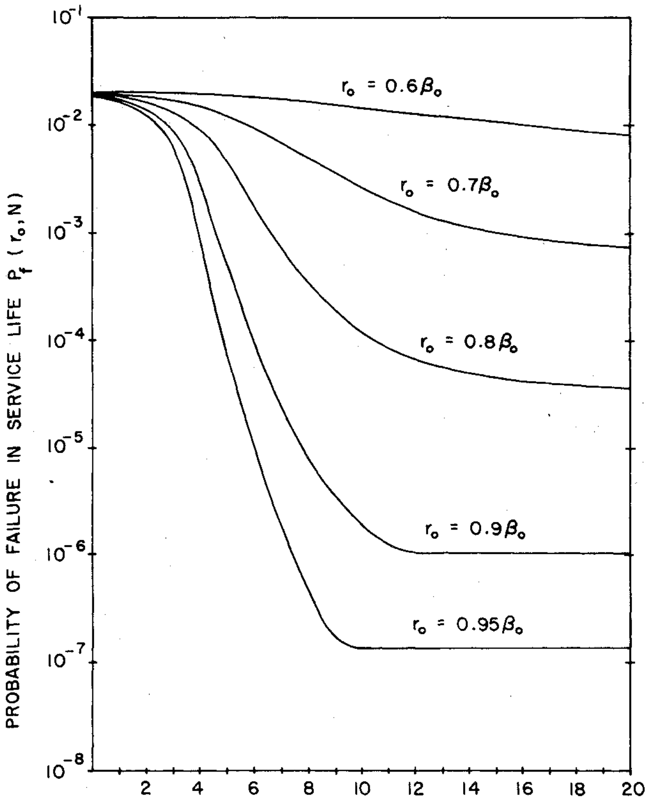


Fig. 5 Probability of failure in design service life T vs number of proof tests N and proof load level r_0.

in practical application. Any improvement beyond Poisson approximation will require the information of the power spectral density of random service stress.[19-22]

For the sake of simplicity, only the reliability analysis of structures under *periodic* proof tests is presented. Proof tests may not be periodic. However, the solution of failure probability under nonperiodic proof tests can be derived in a similar fashion with some modifications without difficulty.

It has been shown that the structural reliability increases as the proof load level increases or as the number of periodic proof tests increases. However, as the number of periodic proof tests increases, the cost of proof tests, including the cost of performing proof tests, down time, nonavailability for service, etc., increases. Furthermore, as the proof load level increases, the number of structures to be destroyed under the proof tests increases, thus increasing the cost of replacement. As a result, there is a tradeoff potential between the structural reliability and costs of proof tests and replacements. A study of this subject for the determination of the optimal periodic proof test will be reported shortly.

The original application of the proof test is for brittle materials (e.g., Refs. 16 and 17). The application of the proof test to materials not exactly brittle requires special considerations. If the material produces a large yield strain after yielding, then the proof load level should not exceed the yield stress. The structure considered herein refers to either the small structural component or the large structure. When a large complex structure with many details is considered for proof testing as a whole, it should be recognized that at some details, such as discontinuities, bolts, holes, etc., yielding may occur locally under proof load due to stress concentration. This, however, cannot be discussed in general, and it has to be considered case by case, depending on the particular structural geometry and complexity. Further study is needed in this area.

Appendix: Statistical Distribution of Residual Strength after Crack Initiation and Survival Probability

The distribution function of the ultimate strength $R(0)$ after the initial (first) proof test prior to service is given by Eq. (9). The ultimate strength $R(0)$ holds without change until a fatigue crack is initiated. Let the fatigue crack be initiated at tth flight hour in the first service interval $(0,T_0)$. Then, the residul strength decreases after crack initiation following Eq. (7). The residual strength at $t+\tau$ flight hours is related to $R(0)=R(t)$ through

$$R^c(t+\tau)=R^c(0)-\Phi\tau \tag{A1}$$

The distribution function of $R(t+\tau)$ can be obtained from that of $R(0)$, Eq. (9), through the transformation of Eq. (A1) as follows:

$$F_{R(t+\tau)}(x)=1-\exp\{(r_0/\beta_0)^{\alpha_0}-([x^c+\Phi\tau]/\beta_0{}^c)^{\alpha_0/c}\} \tag{A2}$$

The distribution function of the residual strength $R(T_0-)$ right before the second proof test performed at T_0 follows from Eq. (A2):

$$F_{R(T_0-)}(x)=1-\exp\{(r_0/\beta_0)^{\alpha_0} - ([x^c+\Phi(T_0-t)]/\beta_0{}^c)^{\alpha_0/c}\} \tag{A3}$$

After the second proof test performed at T_0 flight hours, the distribution function $F_{R(T_0)}(x)$ of the surviving structures can be obtained from $F_{R(T_0-)}(x)$ given by Eq. (A3):

$$F_{R(T_0)}(x)=1-\{P[R(T_0-)>x]/P[R(T_0-)\geq r_0]\};x\geq r_0 \tag{A4}$$

Substitution of Eq. (A3) into Eq. (A4) yields

$$F_{R(T_0)}(x)=1-\exp\left\{\left[\frac{r_0{}^c+\Phi(T_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c} - \left[\frac{x^c+\Phi(T_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}\right\};x\geq r_0 \tag{A5}$$

During the second proof test performed at T_0, the probability of surviving such a proof test, under the condition that the original structure does not fail in service, i.e., $R(t)>S(t)$ for $0\leq t<T_0$, where $S(t)$ is the random service stress, denoted by $B_1^*(t)$, is

$$B_1^*(t)=P[R(T_0-)>r_0]=\exp\{(r_0/\beta_0)^{\alpha_0} - ([r_0{}^c+\Phi(T_0-t)]/\beta_0{}^c)^{\alpha_0/c}\} \tag{A6}$$

Let $R(2T_0-)$ be the residual strength at $2T_0$ before the third proof test. Because of the strength degradation, $R(2T_0-)$ is related to $R(T_0)$ through Eq. (7), $R^c(2T_0-)=R^c(T_0)-\Phi T_0$. Then the distribution function of $R(2T_0-)$ can be obtained from that of $R(T_0)$ given by Eq. (A5) as follows:

$$F_{R(2T_0-)}(x)=1-\exp\left\{\left[\frac{r_0{}^c+\Phi(T_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c} - \left[\frac{x^c+\Phi(2T_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}\right\} \tag{A7}$$

After the third proof test performed at $2T_0$, the distribution function $F_{R(2T_0)}(x)$ for the surviving structures can be obtained from Eq. (A7), in a similar transformation as given by Eq. (A4):

$$F_{R(2T_0)}(x)=1-\exp\left\{\left[\frac{r_0{}^c+\Phi(2T_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}-\left[\frac{x^c+\Phi(2T_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}\right\};x\geq r_0 \tag{A8}$$

The probability of surviving the third proof test performed at $2T_0$, denoted by $\bar{B}_2(t)$, follows from Eq. (A7) as

$$\bar{B}_2(t)=P[R(2T_0-)>r_0]=\exp\left\{\left[\frac{r_0{}^c+\Phi(T_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}-\left[\frac{r_0{}^c+\Phi(2T_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}\right\} \tag{A9}$$

The probability $B_2^*(t)$ of surviving both the second proof test performed at T_0 and the third proof test performed at $2T_0$ is

$$B_2^*(t)=B_1^*(t)\bar{B}_2(t)=\exp\{(r_0/\beta_0)^{\alpha_0}-\{[r_0{}^c+\Phi(2T_0-t)]/\beta_0{}^c\}^{\alpha_0/c}\} \tag{A10}$$

in which Eqs. (A6) and (A9) have been used.

In a similar manner, the distribution function $F_{R(jT_0)}(x)$ of the residual strength *right after* the $j+1$th proof test performed at jT_0, *for the original structure*, can be derived as

$$F_{R(jT_0)}(x)=1-\exp\left\{\left[\frac{r_0{}^c+\Phi(jT_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}-\left[\frac{x^c+\Phi(jT_0-t)}{\beta_0{}^c}\right]^{\alpha_0/c}\right\};x\geq r_0 \tag{A11}$$

and the probability of surviving all of the previous proof tests up to jT_0, denoted by $B_j^*(t)$, under the condition that it does not fail in service, can be shown as

$$B_j^*(t)=\exp\{(r_0/\beta_0)^{\alpha_0}-\{[r_0{}^c+\Phi(jT_0-t)]/\beta_0{}^c\}^{\alpha_0/c}\} \tag{A12}$$

Note that $B_j^*(t)$, $j=1, 2, \ldots$, given by Eqs. (A6, A10, and A12), are conditional probabilities under the condition that the fatigue crack is initiated at t and that the original structure does not fail in service before the $j+1$th proof test.

References

[1]Eggwerts, S., "Investigation of Fatigue Life and Residual Strength of Wing Panel for Reliability Purposes," *Probabilistic Aspects of Fatigue*, American Society for Testing Materials, STP 511, 1972, pp. 75-105.

[2]Davidson, J. R., "Reliability After Inspection," *Fatigue of Composite Materials*, American Society for Testing Materials, STP-569, 1975, pp. 323-334.

[3]Davidson, J. R., "Reliability and Structural Integrity," 10th Anniversary Meeting of the Society of Engineering Science, Raleigh, N. C., 1973.

[4]Freudenthal, A. M., "Reliability Analysis Based on Time to First Failure," Fifth ICAF Symposium, Australia, 1967.

[5]Freudenthal, A. M. and Wang, P. Y., "Ultimate Strength Analysis," Air Force Materials Lab., AFML-TR-69-60, 1969.

[6]Graham, T. W. and Tetelman, A. S., "The Use of Crack Size Distribution and Crack Detection for Determining the Probability of Fatigue Failure," AIAA Paper 74-394, April 1974, Las Vegas, Nev.

[7]Yang, J.-N. and Trapp, W. J., "Reliability Analysis of Aircraft Structures Under Random Loading and Periodic Inspection," *AIAA Journal*, Vol. 12, Dec. 1974, pp. 1623-1630.

[8]Yang, J.-N. and Trapp, W. J., "Inspection Frequency Optimization for Aircraft Structures Based on Reliability Analysis," *Journal of Aircraft*, Vol. 12, May 1975, pp. 494-496.

[9]Yang, J.-N. and Trapp, W. J., "Joint Aircraft Loading/Structure Response Statistics of Time to Service Crack Initiation," *Journal of Aircraft*, Vol. 13, No. 1, April 1976, pp. 270-278.

[10]Yang, J.-N., "Statistical Estimation of Service Cracks and Maintenance Cost for Aircraft Structures," AIAA Paper 75-767, May 1975, Denver, Colo.; accepted to be published in *Journal of Aircraft*.

[11]Whittaker, I. C. and Saunder, S. C., "Application of Reliability Analysis to Aircraft Structures Subjected to Fatigue Crack Growth and Periodic Inspection," Air Force Materials Lab., AFML-TR-73-92, 1973.

[12]Hanagud, S. and Uppaluri, B., "Reliability-Based Optimum Inspection and Maintenance Procedures," *Journal of Aircraft*, Vol. 12, April 1975, pp. 403-410.

[13]Tiffany, C. F., "The Design and Development of Fracture Resistant Structures," *Proceedings of the Colloquium on Structural Reliability*, 1972, Carnegie-Mellon Univ., Pa., pp. 210-215.

[14]Tiffany, C. F., "On the Prevention of Delayed Time Failures of Aerospace Pressure Vessels," *Journal of the Franklin Institute*, Vol. 290, June 1970, pp. 567-582.

[15]Tiffany, C. F., Master, J. N., and Paul, F. A., "Some Fracture Considerations in the Design and Analysis of Spacecraft Pressure Vessels," American Society of Metals, TR-C6-2.3, 1968.

[16]Barnett, R. L. and Hermann, P. C., "Proof Testing in Design with Brittle Materials," *Journal of Spacecraft and Rockets*, Vol. 2, June 1965, pp. 956-961.

[17]Heer, E. and Yang, J.-N., "Structural Optimization Based on Fracture Mechanics and Reliability Criteria," *AIAA Journal*, Vol. 9, May 1971, pp. 621-628.

[18]Shinozuka, M. and Yang, J.-N., "Optimum Structural Design Based on Reliability and Proof Load Test," *Annals of Assurance Science, Proceedings of the Reliability and Maintainability Conference*, Vol. 8, 1969, pp. 375-391.

[19]Lin, Y. K., *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, p. 1067.

[20]Lin, Y. K., "On First-Excursion Failure of Randomly Excited Structures," *AIAA Journal*, Vol. 8, April 1970, pp. 720-725.

[21]Lin, Y. K., "On First-Excursion Failure of Randomly Excited Structures, II," *AIAA Journal*, Vol. 8, Oct. 1970, pp. 1888-1890.

[22]Yang, J.-N. and Shinozuka, M., "On First-Excursion Probability in Stationary Narrow-Band Random Vibration, II," *Journal of Applied Mechanics*, Vol. 39, March 1972, pp. 733-738.

[23]Snider, H. L., et al., "Residual Strength and Crack Propagation Test on C-130 Airplane Center Wings with Service-Imposed Fatigue Damage," NASA CR-2075, 1972.

[24]Wood, H. A., "The Use of Fracture Mechanics Principles in the Design and Analysis of Damage Tolerant Aircraft Structures," *Fatigue Life Prediction for Aircraft Structures and Materials*, AGARD-LS-62, 1973, pp. 4.1-4.3.

[25]Whaley, R. E., et al., "Fatigue Crack Propagation and Residual-Static-Strength Results on Full-Scale Transport Airplane Wings," NACA TN-3847, 1956.

[26]Houbolt, J. C., Steiner, R. and Platt, K. G., "Dynamic Response of Airplanes to Atmospheric Turbulence Including Flight Data on Input and Response," NACA Rept. R-199, 1964.

[27]"Airplane Strength and Rigidity, Flight Loads," Military Specification MIL-A-008861A, U. S. Air Force, Washington, D. C., 1971.

[28]Neulieb, R. L., Garrison, J. N., and Golden, D. J., "Atmospheric Turbulence Field Parameters Determination," Air Force Flight Dynamic Lab., AFFDL-TR-72-51, 1972.

[29]Celata, R. T., "Mission Analysis to Determine Maneuver Load Factor Exceedance Spectra," General Dynamics Rept. FZM-12-10783, 1969.

[30]Lybold, H. A., "Simulation of Flight Maneuver Load Distribution by Utilizing Non-Gaussian Random Load History," NASA TND-6570, 1971.

[31]Press, H., Meadow, M. T., and Hadlock, I., "A Re-evaluation of Data on Atmospheric Turbulence and Airplane Gust Loads for Application in Spectral Calculation," NASA Rept. 1272, 1956.